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On crack perturbation in thermoelastic media

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Abstract

Explicit asymptotic model is presented for a singularly perturbed boundary value problem of the uncoupled thermo-elasticity. Interaction of a crack and a small inclusion with different elastic and thermo-elastic moduli is analyzed. Asymptotical formulae are derived for the crack trajectories in terms of Pólya-Szegö tensors associated with the defects. Effect of the temperature on the deflection of the crack is compared with propagation of the crack in heterogeneous elastic solid with zero temperature increment. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Fracture propagation in inhomogeneous media is one of the subjects which are of the interest in the research community in the recent years. Extensive work has been published on the fracture of brittle materials and the crack formation (see monographs by Sih, 1978; Sih and Chen, 1981; Sih, 1991). Starting from Bueckner (1970) the technique of the weight functions is used to evaluate the stress intensity factors for the problems with cracks. We would like to refer to the papers by Willis and Movchan (1995) and Movchan and Willis (1995) who derived the weight functions for 3D cracks and analyzed three-dimensional propagating crack (Willis and Movchan, 1997). The question of crack propagation in inhomogeneous elastic media has been studied by Movchan et al. (1991), interaction with particular inclusions has been considered by Bigoni et al. (1998). Bigoni et al. (1996) compared the results of asymptotical analysis with the experimental data obtained in Ceramic Center (Bologna). The crack-inclusion interaction has been studied by Rubinstein (1986) with the use of the integral equation

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technique. The interaction of a crack with an array of micro-cracks has been analyzed by Rubinstein and Choi (1988).

In the text below, we address the problem of interaction between defects and a growing crack in the case of thermo-elastic medium. It is shown that the effect of fracture propagation significantly depends on the temperature contribution which should be taken into account. Comparison of the crack propagation in elastic (no thermal stresses) and thermo-elastic media is given. An asymptotic method is applied to obtain explicit formulae for the stress intensity factors and the crack trajectory. The criterion of Sih (1991) is used to characterize the direction of the crack propagating in a brittle elastic material. The Pólya-Szegö tensors are employed on the final stage of the algorithm to obtain analytical formulae for the crack deflection.

The mathematical model presented in this paper uses just one of the well-known criteria for the orientation of the crack in a brittle material. However, the model allows for straightforward modifications involving other criteria of fracture. The fracture criterion is used only at the final step of the asymptotic algorithm (see formula (22) for the crack deflection). Comparison of different fracture criteria is not covered in the present text.

We consider separately the elastic deflection (deflection corresponding to zero increment of the temperature) and the thermal deflection (difference between total deflection and the elastic one). In particular, the conditions are analyzed when the thermal deflection and the elastic deflection compensate each other and the temperature causes the reduction in the amplitude of the deflection. This problem can be regarded as the problem of the residual stress effect in the composite medium. Thermal stresses in the inclusion produce a jump in the displacement field on the interface boundary. Jump conditions of the same kind correspond to the residual stresses occurring under cooling. The problem can be reformulated to be a problem of interaction between a crack and an inclusion with non-perfect interface. The interface is specified by the jump, which is the function of the temperature, either in displacement or in traction boundary conditions.

2. Mathematical formulation of the problem

Consider an infinite thermoelastic plane with a small inclusion G_ε and a crack M_0. Let the elastic material of the plane be characterized by the Lamé constants μ and λ and the thermoelastic constant γ. The inclusion G_ε is characterized by μ_0, λ_0 and γ_0.

On crack faces $M_0^\pm$ the tractions are specified, on the boundary ∂G_ε the interface boundary conditions are imposed. We solve the following boundary value problem of the linear uncoupled thermo-elasticity. The equilibrium equations are specified in the infinite plane with an elastic inclusion:

$$\boldsymbol{L}(\boldsymbol{u};\boldsymbol{x}) := \mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} = \gamma\nabla T(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^2\backslash\{G_\varepsilon\cup M_0\},$$

$$\boldsymbol{L}_0(\boldsymbol{u}_0;\boldsymbol{x}) := \mu_0\nabla^2\boldsymbol{u}_0 + \left(\lambda_0+\mu_0\right)\nabla\nabla\cdot\boldsymbol{u}_0 = \gamma_0\nabla T(\boldsymbol{x}), \quad \boldsymbol{x}\in G_\varepsilon. \tag{1}$$

An ideal contact is prescribed at the interface between the inclusion and the matrix:

$$\boldsymbol{\sigma}^{(n)}(\boldsymbol{u};\boldsymbol{x}) - \gamma T\boldsymbol{n} = \boldsymbol{\sigma}_0^{(n)}(\boldsymbol{u}_0;\boldsymbol{x}) - \gamma_0 T\boldsymbol{n}, \quad \boldsymbol{u}=\boldsymbol{u}_0, \quad \boldsymbol{x}\in\partial G_\varepsilon, \tag{2}$$

where

$$\boldsymbol{\sigma}^{(n)}(\boldsymbol{u};\boldsymbol{x}) = \begin{pmatrix} (2\mu+\lambda)\dfrac{\partial u_1}{\partial x_1} + \lambda\dfrac{\partial u_2}{\partial x_2} & \mu\left(\dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_1}{\partial x_2}\right) \\ \mu\left(\dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_1}{\partial x_2}\right) & (2\mu+\lambda)\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_1}{\partial x_1} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{3}$$

is the traction vector calculated via the elastic displacements only (no temperature change). The thermal tractions $\gamma T\boldsymbol{n}$ always act in the direction of normal vector $\boldsymbol{n}$.

The condition (2) corresponds to the perfect interface boundary conditions in heterogeneous materials: the ‘thermal’ tractions (elastic and thermal forces both) and the displacement vectors are continuous through the interface, while the ‘elastic’ tractions (Eq. (3)) have a jump.

On the crack faces the inhomogeneous traction conditions are specified

$$\boldsymbol{\sigma}^{(n)}(\boldsymbol{u};\boldsymbol{x}) = \boldsymbol{p}(\boldsymbol{x}) + \gamma T\boldsymbol{n}, \quad \boldsymbol{x} \in M_0^{\pm}, \tag{4}$$

and at infinity

$$\boldsymbol{u} \to \boldsymbol{0}, \quad \text{as } |\boldsymbol{x}| \to \infty.$$

Under these conditions the crack propagates due to the forces $\boldsymbol{p}(\boldsymbol{x})$ applied to the crack faces. We can alternatively modify the conditions at infinity on the square-root grows and put $\boldsymbol{p}(\boldsymbol{x})$ in Eq. (4) to be zero.

$$\boldsymbol{u} \to K_{\mathrm{I}}^{\infty}|\boldsymbol{x}|^{1/2}\Phi^{I}, \quad \text{as } |\boldsymbol{x}| \to \infty, \tag{5}$$

where K_{I}^{∞} is the stress intensity factor taking into account the effect of all forces on the crack faces and Φ^I is the Mode-I Williams vector.

In addition, the following heat conductivity problem is considered for the temperature field $T(\boldsymbol{x})$:

$$k\nabla^2 T(\boldsymbol{x}) = w(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2 \backslash G_\varepsilon, \quad k_0\nabla^2 T_0(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in G_\varepsilon,$$

$$k\frac{\partial T}{\partial n} = k_0\frac{\partial T_0}{\partial n}, \quad T = T_0, \quad \boldsymbol{x} \in \partial G_\varepsilon,$$

$$T|_{M_0^+} = T|_{M_0^-}, \quad \partial_n T|_{M_0^+} + \partial_n T|_{M_0^-} = 0,$$

$$T \to T_\infty(x), \quad |\boldsymbol{x}| \to \infty,$$

where $\boldsymbol{n}$ is a unit outward normal vector, k and k_0 are the thermal conductivities of both phases, $w(\boldsymbol{x})$ is the intensity of the heat sources.

In the text below, the solution of the inverse problem — defining the trajectory of crack propagation in an inhomogeneous material is presented. We begin with the formal asymptotic procedure where the solution is specified in terms of series in powers of the parameter ε (ratio between the effective radius of the inclusion and the distance between the crack and the defect). As a result the increment of the stress intensity factor can be found as a function of the Pólya-Szegö matrix of the inclusion. Then we apply the criterion of the quasi-static crack propagation (Sih and Chen, 1981). This criterion has been derived by Sih from the analysis of the strain energy density around the crack. Of course, it does not take into account the change in the microstructure of material near the crack tip. The strain gradient effects, formation of the plastic zones around the crack tip and non-elastic dissipation of the energy are not considered. This criterion can be applied to evaluate the integral propagation of the crack when the deflection angle is much less than the angle associated with the material length scale and the time of propagation is much bigger then the relaxation time in the current material. The crack is supposed to be a Mode-I crack (no shear forces on the crack phases). Different criteria can be used (see discussion in Sih, 1991), but analytical formulae for the crack deflection function become more complicated in that case.

For illustration we give the contour plot of the typical stress distribution around the crack tip caused by the presence of inhomogeneity in homogeneous external Mode-I loading (Fig. 1). The finite-element modelling has been carried out with the use of COSMOS/M finite element package under the license of the University of Bath (UK).

3. Formal asymptotics

Assuming that $\mathrm{diam} G_\varepsilon \ll \mathrm{dist}\{G_\varepsilon, M_0\}$, we introduce a small parameter as follows

$$\varepsilon = \frac{\mathrm{diam} G_\varepsilon}{\mathrm{dist}\{G_\varepsilon, M_0\}}.$$

First, we consider the temperature distribution in a plane with an inclusion. In the case of equal thermal conductivities ($k = k_0$) the thermal boundary layers do not occur near the inclusion and the temperature field can be found as a solution of the following problem:

$$k\nabla^2 T(\boldsymbol{x}) = w(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2,$$

$$T \to T_\infty, \quad |\boldsymbol{x}| \to \infty. \tag{6}$$

In the case of different thermal conductivities of the inclusion and matrix in the presence of heat sources, we shall construct the boundary layer and apply the asymptotic series expansion:

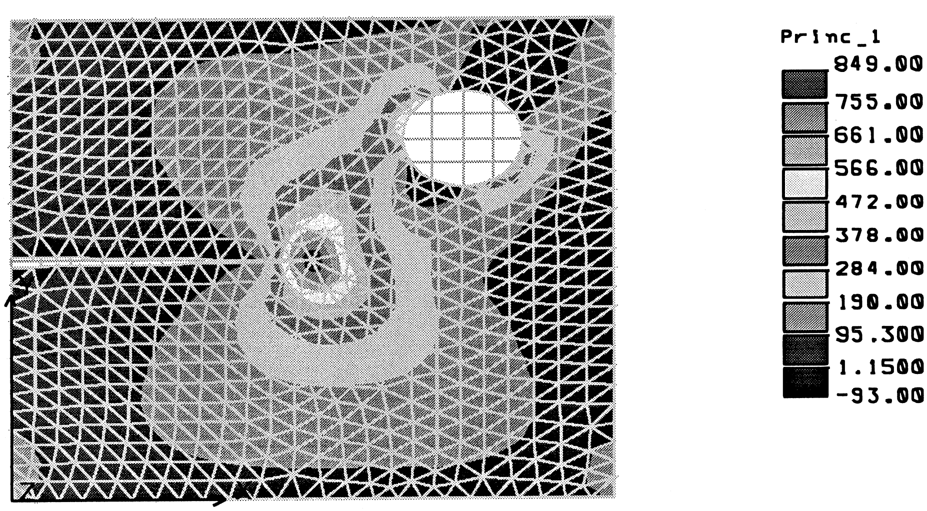

Fig. 1. Mode-I crack in an elastic plane with a circular cavity: distribution of the maximal eigenvalue of the stress tensor.

$$T(\boldsymbol{x}) = T^{(0)}(\boldsymbol{x}) + \varepsilon T^{(1)}(\boldsymbol{X}) + O(\varepsilon^2), \tag{7}$$

where $T^{(1)}(\boldsymbol{X})$ is a boundary layer solution which compensates the discrepancy in the interface boundary conditions produced by $T^{(0)}(\boldsymbol{x})$.

Given the temperature field (Eq. (7)) we seek the displacement vector $\boldsymbol{u}$ as a solution of the boundary value problem (1)–(4) in the form

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^{(0)}(\boldsymbol{x}) + \varepsilon\, \boldsymbol{u}^{(1)}(\boldsymbol{X}) + \varepsilon^2 \boldsymbol{u}^{(2)}(\boldsymbol{x}) + O(\varepsilon^3). \tag{8}$$

3.1. The leading-order term of the displacement field $\boldsymbol{u}^{(0)}$

Here, *the leading order* term $\boldsymbol{u}^{(0)}(\boldsymbol{x})$ is a solution of the boundary value problem in $\mathbb{R}^2 \backslash M_0$ (without the defect):

$$\mu \nabla^2 \boldsymbol{u}^{(0)}(\boldsymbol{x}) + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u}^{(0)}(\boldsymbol{x}) = \gamma \nabla_x T^{(0)} + \gamma \nabla_X T^{(1)}, \quad \boldsymbol{x} \in \mathbb{R}^2 \backslash M_0,$$

$$\boldsymbol{\sigma}^{(n)}\left(\boldsymbol{u}^{(0)}; \boldsymbol{x}\right) = \boldsymbol{p}(\boldsymbol{x}) + \gamma T(\boldsymbol{x})\boldsymbol{n}, \quad \boldsymbol{x} \in M_0^{\pm}. \tag{9}$$

We introduce the following linearly independent vector functions which satisfy the homogeneous Lamé system written in the stretched variables $\boldsymbol{X} = \varepsilon^{-1}\boldsymbol{x}$

$$\boldsymbol{U}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{U}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\boldsymbol{V}^{(1)} = \begin{pmatrix} X_1 \\ 0 \end{pmatrix}, \quad \boldsymbol{V}^{(2)} = \begin{pmatrix} 0 \\ X_2 \end{pmatrix}, \quad \boldsymbol{V}^{(3)} = \begin{pmatrix} X_2 \\ X_1 \end{pmatrix}, \quad \boldsymbol{V}^{(4)} = \begin{pmatrix} -X_2 \\ X_1 \end{pmatrix}.$$

The leading term $\boldsymbol{u}^{(0)}$ of the expansion (8) admits the following representation in the vicinity of the inclusion (coordinates $\boldsymbol{x}$ are the Cartesian coordinates with the origin at the center of the inclusion):

$$\boldsymbol{u}^{(0)}(\boldsymbol{x}) \sim u^{(0)}(\boldsymbol{0}) + \varepsilon X_1 \left.\frac{\partial \boldsymbol{u}^{(0)}(\boldsymbol{x})}{\partial x_1}\right|_0 + \varepsilon \boldsymbol{X}_2 \left.\frac{\partial \boldsymbol{u}^{(0)}(\boldsymbol{x})}{\partial x_2}\right|_0$$

$$= A_1 \boldsymbol{U}^{(1)}(\boldsymbol{X}) + A_2 \boldsymbol{U}^{(2)}(\boldsymbol{X}) + C_1 \varepsilon \boldsymbol{V}^{(1)}(\boldsymbol{X}) + C_2 \varepsilon \boldsymbol{V}^{(2)}(\boldsymbol{X}) + C_3 \varepsilon \boldsymbol{V}^{(3)}(\boldsymbol{X}) + C_4 \varepsilon \boldsymbol{V}^{(4)}(\boldsymbol{X}), \tag{10}$$

where A_i, C_k are constants defined in terms of the components of the strain tensor evaluated for the leading term of the displacement field

$$C_1 = \frac{\partial u_1^{(0)}}{\partial x_1} = \varepsilon_{11}(\boldsymbol{u}^{(0)}), \quad C_2 = \frac{\partial u_2^{(0)}}{\partial x_2} = \varepsilon_{22}(\boldsymbol{u}^{(0)}), \quad C_3 = \frac{1}{2}\left\{\frac{\partial u_2^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_1}\right\} = \varepsilon_{12}(\boldsymbol{u}^{(0)}).$$

3.2. The boundary layer solution $\varepsilon\, \boldsymbol{u}^{(1)}$

The second term of the expansion, Eq. (8), can be specified as a solution of the following boundary value problem written in the stretched coordinates:

$$\mu\nabla^2\boldsymbol{u}^{(1)}(\boldsymbol{X}) + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}^{(1)}(\boldsymbol{X}) = \boldsymbol{0}, \quad \boldsymbol{X}\in\mathbb{R}^2\backslash G,$$

$$\mu_0\nabla^2\boldsymbol{u}_0^{(1)}(\boldsymbol{X}) + (\lambda_0+\mu_0)\nabla\nabla\cdot\boldsymbol{u}_0^{(1)}(\boldsymbol{X}) = \boldsymbol{0}, \quad \boldsymbol{X}\in G,$$

$$\boldsymbol{\sigma}^{(n)}\left(\boldsymbol{u}^{(1)};\boldsymbol{X}\right) - \boldsymbol{\sigma}_0^{(n)}\left(\boldsymbol{u}_0^{(1)};\boldsymbol{X}\right) = \boldsymbol{\sigma}_0^{(n)}\left(\boldsymbol{u}_0^{(0)};\boldsymbol{X}\right) - \boldsymbol{\sigma}^{(n)}\left(\boldsymbol{u}^{(0)};\boldsymbol{X}\right) + (\gamma-\gamma_0)T(\boldsymbol{X}) \quad \boldsymbol{X}\in\partial G,$$

$$\boldsymbol{u}(\boldsymbol{X}) = \boldsymbol{u}_0(\boldsymbol{X}), \quad x\in\partial G, \quad \boldsymbol{\sigma}^{(n)}\left(\boldsymbol{u}^{(1)};\boldsymbol{X}\right)\to\boldsymbol{0}, \quad |\boldsymbol{X}|\to\infty \tag{11}$$

where we substitute the expression (10) instead of the leading term $\boldsymbol{u}^{(0)}$.

The field $\varepsilon\,\boldsymbol{u}^{(1)}(\boldsymbol{X})$ is a boundary layer characterizing the changes in the stress components near the defect. If we look at the interface traction conditions of the problem (11) we can note that the traction jump is defined by the leading term of the displacement field $\boldsymbol{u}^{(0)}$ and by the temperature field $T(\boldsymbol{X})$. Since both terms are uncoupled, we can split the jump in the traction boundary conditions (11) associated with the term $\boldsymbol{u}^{(0)}$ and with the temperature. The solution $\boldsymbol{u}^{(1)}$ can be represented as a linear combination of a solution with an elastic jump and a solution due to the temperature.

3.3. Model dipole fields

Note that stresses produced by the rigid body displacement $\boldsymbol{U}^{(1)}$, $\boldsymbol{U}^{(2)}$ and $\boldsymbol{V}^{(4)}$ are equal to zero. For other vector polynomials $\boldsymbol{V}^{(i)}$, $i = 1, 2, 3$ we construct the field $\boldsymbol{W}^{(i)}$ which compensates the discrepancy left by $\boldsymbol{V}^{(i)}$ in the interface boundary conditions. At infinity the vector functions $\boldsymbol{W}^{(i)}$ admit the asymptotic representation (see Movchan and Serkov, 1997):

$$\boldsymbol{W}^{(i)}(\boldsymbol{X}) = \sum_{k=1}^{3}\mathscr{P}_{ik}\mathfrak{D}^{(k)}\boldsymbol{T}(\boldsymbol{X}) + O\left(|\boldsymbol{X}^{-2}|\right), \tag{12}$$

where $\mathscr{P}_{ik}$ are the components of the Pólya-Szegö matrix of the defect, $\boldsymbol{T}(\boldsymbol{X})$ is the Green's tensor and $\mathfrak{D}^{(k)} \equiv \boldsymbol{V}^{(k)}(\partial/\partial\boldsymbol{X})$ are vector differential operators associated with vectors $\boldsymbol{V}^{(k)}(\boldsymbol{X})$.

The thermo-elastic field $\boldsymbol{W}^{(4)}$ solves the problem:

$$\mu\nabla^2\boldsymbol{W}^{(4)}(\boldsymbol{X}) + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{W}^{(4)}(\boldsymbol{X}) = \boldsymbol{0}, \quad \boldsymbol{X}\in\mathbb{R}^2\backslash G,$$

$$\mu_0\nabla^2\boldsymbol{W}_0^{(4)}(\boldsymbol{X}) + (\lambda_0+\mu_0)\nabla\nabla\cdot\boldsymbol{W}_0^{(4)}(\boldsymbol{X}) = \boldsymbol{0}, \quad \boldsymbol{X}\in G,$$

$$\boldsymbol{\sigma}^{(n)}\left(\boldsymbol{W}^{(4)};\boldsymbol{X}\right) - \boldsymbol{\sigma}_0^{(n)}\left(\boldsymbol{W}_0^{(4)};\boldsymbol{X}\right) = (\lambda-\lambda_0)T^{(0)}\boldsymbol{n}, \quad \boldsymbol{W}^{(4)} = \boldsymbol{W}_0^{(4)}, \quad \boldsymbol{X}\in\partial G,$$

$$\boldsymbol{\sigma}^{(n)}\left(\boldsymbol{W}^{(4)};\boldsymbol{X}\right)\to\boldsymbol{0}, \quad |\boldsymbol{X}|\to\infty \tag{13}$$

It admits the following asymptotic representation as $|\boldsymbol{X}|\to\infty$

$$\boldsymbol{W}^{(4)}(\boldsymbol{X}) = \sum_{k=1}^{3}\mathscr{D}_k\mathfrak{D}^{(k)}\boldsymbol{T}(\boldsymbol{X}) + O\left(|\boldsymbol{X}|^{-2}\right). \tag{14}$$

In Eq. (14) $\mathscr{D}_k$ are constants uniquely defined by the thermo-elastic properties and geometry of the defect (compare the representation (14) with the asymptotic expansions of the fields $\boldsymbol{W}^{(i)}(\boldsymbol{X})$, $i = 1, 2, 3$ (Eq. (12))).

Using the notations above, the second asymptotic term can be written in the form:

$$\boldsymbol{u}^{(1)}(\boldsymbol{X}) = \sum_{i=1}^{3} C_i \boldsymbol{W}^{(i)}(\boldsymbol{X}) + \boldsymbol{W}^{(4)}(\boldsymbol{X}). \tag{15}$$

3.4. The third term of asymptotic expansion $\varepsilon^2 \boldsymbol{u}^{(2)}$

The third term $\varepsilon^2 \boldsymbol{u}^{(2)}(\boldsymbol{x})$ compensates the discrepancy in the boundary conditions on the crack surfaces $M_0^{\pm}$:

$$\mu \nabla^2 \boldsymbol{u}^{(2)}(\boldsymbol{x}) + (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u}^{(2)}(\boldsymbol{x}) = \boldsymbol{0}, \quad \boldsymbol{x} \in \mathbb{R}^2 \backslash M_0,$$

$$\boldsymbol{\sigma}^{(n)}\left(\boldsymbol{u}^{(2)}; \boldsymbol{x}\right) = -\boldsymbol{\sigma}^{(n)}\left(\sum_{k=1}^{3} \sum_{i=1}^{3} C_i \mathscr{P}_{ik} + \mathscr{D}_k \mathfrak{D}^{(k)} \boldsymbol{T}(\boldsymbol{x}); \boldsymbol{x} \right), \quad \boldsymbol{x} \in M_0^{\pm}, \tag{16}$$

and can be represented in the form:

$$\boldsymbol{u}^{(2)}(\boldsymbol{x}) = \sum_{i,\, k=1}^{3} \mathscr{P}_{ik} \left(\boldsymbol{T}^{(k)}(\boldsymbol{x}) - \mathfrak{D}^{(k)} \boldsymbol{T}(\boldsymbol{x}) \right) C_n + \sum_{k=1}^{3} \mathscr{D}_k \left(\boldsymbol{T}^{(k)}(\boldsymbol{x}) - \mathfrak{D}^{(k)} \boldsymbol{T}(\boldsymbol{x}) \right). \tag{17}$$

The field $\boldsymbol{T}^{(k)}$ can be found as a solution of the Lamé system with the dipole body forces acting at the center of the small defect and zero tractions on the crack surfaces:

$$\mu \nabla^2 \boldsymbol{T}^{(k)}(\boldsymbol{x}) + (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{T}^{(k)}(\boldsymbol{x}) + \mathfrak{D}^{(k)} \delta(\boldsymbol{x}) = \boldsymbol{0}, \quad \boldsymbol{x} \in \mathbb{R}^2 \backslash M_0, \quad \boldsymbol{\sigma}^{(n)}\left(\boldsymbol{T}^{(k)}; \boldsymbol{x}\right) = \boldsymbol{0}, \quad \boldsymbol{x} \in M_0^{\pm}. \tag{18}$$

4. The stress intensity factors and the crack trajectory

Following Bueckner (1970) the stress intensity factors can be computed in terms of tractions applied to the faces of a crack. He introduced the weight functions $\boldsymbol{\xi}^{\mathrm{I}}$, $\boldsymbol{\xi}^{\mathrm{II}}$ such that the stress intensity factors K_{I} and K_{II} corresponding to Mode-I and Mode-II loading can be represented via integral identities

$$K_{\mathrm{I}} = \frac{\sqrt{2}}{\pi} \int^{M_0} \boldsymbol{p}(\boldsymbol{x}) \boldsymbol{\xi}^{\mathrm{I}}(\boldsymbol{x})\, \mathrm{d}s, \quad K_{\mathrm{II}} = \frac{\sqrt{2}}{\pi} \int^{M_0} \boldsymbol{p}(\boldsymbol{x}) \boldsymbol{\xi}^{\mathrm{II}}(\boldsymbol{x})\, \mathrm{d}s,$$

where $\boldsymbol{p}(\boldsymbol{x})$ are tractions applied to crack faces, $\boldsymbol{\xi}^{\mathrm{I}}$ is the weight function for Mode-I loading, $\boldsymbol{\xi}^{\mathrm{II}}$ is the weight function for Mode-II loading. The similar idea has been used by Maz'ya et al. (1992). Their method uses the Betti's formula (Timoshenko and Goodier, 1951) to obtain representation for the stress intensity factor.

In this section our intention is to define the increment of the stress intensity factor due to the presence of the inclusion. Thus, we apply the Betti's formula to the weight function $\boldsymbol{\xi}^{\mathrm{II}}$ and the second asymptotic term $\boldsymbol{u}^{(2)}$. The leading term $\boldsymbol{u}^{(0)}$ of the asymptotic expansion, Eq. (8), gives us the stress intensity factor but not the increment. First asymptotic term $\boldsymbol{u}^{(1)}$ in Eq. (8) is the boundary layer near the defect and it vanishes far away from the inclusion (on the crack faces). The next asymptotic term which describes the difference in fields with and without inclusion is $\boldsymbol{u}^{(2)}$. Thus if we would like to obtain the increment of the stress intensity factor we should take into account $\boldsymbol{u}^{(2)}$. The Betti's formula is applied in the ring $\Xi_R = \{\boldsymbol{x}: \frac{1}{R} \le |\boldsymbol{x} - (l,\ 0)| \le R\}$. Integration goes along closed contour which includes the circle C_1 around the neighborhood of perturbed crack, upper and lower faces of crack$M_0^{\pm}$ and the circle C_2 with radius R, which takes into account the far fields. The limit when R tends to infinity is considered only after integration — we apply the Betti's formula in the bounded domain.

$$\int_{\Xi_R} \left\{ \xi^{\mathrm{II}}(\boldsymbol{x}) \cdot \boldsymbol{L}\left(\boldsymbol{u}^{(2)}; \boldsymbol{x}\right) - \boldsymbol{u}^{(2)} \cdot \boldsymbol{L}\left(\xi^{\mathrm{II}}; \boldsymbol{x}\right) \right\} \mathrm{d}x$$

$$= \int_{C_1 \cup M_0^{\pm} \cup C_2} \left\{ \xi^{\mathrm{II}}(\mathrm{x}) \cdot \boldsymbol{\sigma}^{(n)}\left(\boldsymbol{u}^{(2)}; \boldsymbol{x}\right) - \boldsymbol{u}^{(2)} \cdot \boldsymbol{\sigma}^{(n)}\left(\xi^{\mathrm{II}}; \boldsymbol{x}\right) \right\} \mathrm{d}s. \tag{19}$$

Here $\boldsymbol{L}(\cdot, \boldsymbol{x})$ is the Lamé operator of linear elasticity, given in Eq. (1) $\boldsymbol{\sigma}^{(n)}(\cdot, \boldsymbol{x})$ is the operator of boundary conditions (3). The Mode-II weight function is given in terms of the angular part of the Williams Mode-I vector Φ^{I}:

$$\xi^{\mathrm{II}} = \frac{4\mu}{1 + æ} \frac{\partial}{\partial x_2} (r^{1/2} \boldsymbol{\Phi}^{\mathrm{I}}). \tag{20}$$

It is also important to note that while the second term of the asymptotic expansion $\boldsymbol{u}^{(2)}$ satisfies the boundary value problem (16) in whole domain Ξ_R, it admits the asymptotic expansion (17) on the boundary of outer circle C_2 and the asymptotic expansion given by Eq. (9) from Movchan et al. (1991) at the vicinity of the crack tip on the boundary C_1. The laborious calculations show that left side of the Eq. (19) can be evaluated by the expression

$$I = \sum_{i,\ k=1}^{3} \mathscr{P}_{ik} \mathfrak{D}^{(i)} \boldsymbol{u}^{(0)}(\boldsymbol{x}) \mathfrak{D}^{(k)} \xi^{\mathrm{II}}(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{x}_0} + \sum_{k=1}^{3} \mathscr{D}_k \mathfrak{D}^{(k)} \xi^{\mathrm{II}}(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{x}_0}, \quad \boldsymbol{u}^{(0)}(\boldsymbol{x}) = K_{\mathrm{I}} r^{1/2} \boldsymbol{\Phi}^{\mathrm{I}}(\phi) \tag{21}$$

corresponding to the leading term of the asymptotic expansion for the displacement field in the vicinity of the Mode-I crack.

The right side is correspondingly specified in terms of stress intensity factors K_{I} and increment of K_{II} (note that K_{II} for unperturbed crack is supposed to be zero):

$$\frac{1}{2} h'(l) K_{\mathrm{I}}(l) - K_{\mathrm{II}}(l).$$

For further simplifications it is convenient to introduce the auxiliary vector $\mathscr{L}$ with the components:

$$\mathscr{L} = \begin{pmatrix} \dfrac{1}{8\mu\sqrt{2\pi}} \left(\cos\dfrac{5\phi}{2} + (2æ - 3)\cos\dfrac{\phi}{2} \right) \\ \dfrac{1}{8\mu\sqrt{2\pi}} \left(-\cos\dfrac{5\phi}{2} + (2æ - 1)\cos\dfrac{\phi}{2} \right) \\ \dfrac{1}{8\mu\sqrt{x}} \left(\sin\dfrac{5\phi}{2} - \sin\dfrac{\phi}{2} \right) \end{pmatrix},$$

where ϕ denotes the angle between the x-axis and a line joining the crack tip and the center of the inclusion. Using the identity $\mathfrak{D}^{(k)}(r^{1/2}\boldsymbol{\Phi}^{\mathrm{I}})|_{x_0} = \mathscr{L}^{(k)} r^{-1/2}$ and also the expression (20) one can show that the expression (21) (left-hand side of the Betti's formula (19)) can be rewritten in following way

$$I = \frac{2\mu K_{\mathrm{I}}}{(1+æ)r^2} \sum_{i,\,k=1}^{3} \frac{\partial}{\partial\phi}\Big(\cos\phi\,\mathscr{L}^{(i)}\mathscr{P}_{ik}\mathscr{L}^{(k)}\Big) + \frac{4\mu}{(1+æ)r^{3/2}} \sum_{k=1}^{3} \mathscr{D}_k\left(\cos\phi\frac{\partial}{\partial\phi}\mathscr{L}^{(k)} - \frac{1}{2}\sin\phi\,\mathscr{L}^{(k)}\right).$$

Now we take into account Sih (1991) criterion of the crack propagation (the validity of this criterion is discussed in 2):

The crack propagates if the stress intensity factor K_{I} is greater than the critical value K_{I}^{c}. It propagates as pure Mode-I crack, i.e. the stress intensity factor K_{II} is equal to zero.

The crack path deflection is obtained after the integration of the expression (3) in the following form:

$$h(l) = \frac{4\mu}{(1+æ)y_0} \sum_{i,\,k=1}^{3} \Big(\cos\phi\,\mathscr{L}^{(i)}\mathscr{P}_{ik}\mathscr{L}^{(k)}\Big)\Big|_0^{\theta}$$

$$+\frac{8\mu}{K_{\mathrm{I}}(1+æ)\sqrt{y_0}} \sum_{k=1}^{3} \int_0^{\theta} \mathscr{D}_{(k)}\left(\cos\phi\frac{\partial}{\partial\phi}\mathscr{L}^{(k)} - \frac{1}{2}\sin\phi\,\mathscr{L}^{(k)}\right)\frac{\mathrm{d}\phi}{\sqrt{\sin\phi}}, \tag{22}$$

where θ denotes the angle between the x-axis and the line drawn through the crack tip and the center of the inclusion, $h(l)$ is the crack deflection about the x-axis. Note that the formula (22) holds for thermoelastic inhomogeneities of an arbitrary shape.

5. Several examples of fracture propagation

5.1. Interaction between a crack and circular thermoelastic inhomogeneities

The formula (4) can be regarded as the crack deflection for any inclusion or their combination. The morphology of the defects is specified in terms of the Pólya-Szegö matrix $\mathscr{P}$ and the thermo-elastic vector $\mathscr{D}$. The only restriction on the application of this formula is the non-interactive behavior of inclusions. In other words, we suppose that the inclusion is located far away from the crack and its diameter is small in comparison with the distance to the crack. If there are several inclusions in a plane, then the dilute limit is required. In this section we consider an interaction of a crack with a circular inclusion: this is the simplest example, but it allows one the determine the main features of the interaction mechanism.

In the problem where the inclusion is regarded to be circular we need to know the coefficients of matrix $\mathscr{P}$ and vector $\mathscr{D}$. The complex variables technique developed by Muskhelishvili (1963) can been used for these purposes. We omit the technical calculations and rewrite the Pólya-Szegö matrix for a circular inclusion as it is given by Zorin et al. (1988). It is convenient to use the normalized form of the matrix, where R (radius of the inclusion) is supposed to be equal to one:

$$\mathscr{P} = \mu\pi(1+æ)\begin{pmatrix} \Xi+\Theta & \Xi-\Theta & 0 \\ \Xi-\Theta & \Xi+\Theta & 0 \\ 0 & 0 & 2\Theta \end{pmatrix}, \tag{23}$$

where

$$\Theta = \frac{\mu_0 - \mu}{æ\mu_0 + \mu}, \quad \Xi = \frac{2\mu_0(æ - 1) - 2\mu(æ_0 - 1)}{(æ - 1)^2\big(\mu(æ_0 - 1) + 2\mu_0\big)}.$$

To calculate the thermoelastic vector $\mathscr{D}$, we start with the analysis of the boundary conditions of the problem (13) written in terms of the complex potentials

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = \varphi_0(z) + z\overline{\varphi_0'(z)} + \overline{\psi_0(z)} + (\gamma - \gamma_0)\Delta T z,$$

$$\big[æ\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\big]\frac{\mu_0}{\mu} = æ_0\varphi_0(z) - z\overline{\varphi_0'(z)} - \overline{\psi_0(z)}, \tag{24}$$

and look for a solution which decays at infinity like $O(|z|^{-1})$. Due to the Eshelby's theorem (Eshelby, 1957) the solution is linear inside the inclusion. Note that here and further in the text we write the increment of temperature ΔT instead of T. This value can be either positive or negative and depends on initial temperature which corresponds to zero thermal stresses. The complex potentials $\varphi(z)$ and $\psi(z)$ obtained from the analysis of the boundary integral equations based on the boundary conditions (24) have the following form:

$$\varphi(z) = O\left(\frac{1}{|z|^2}\right), \quad \psi(z) = \frac{(\gamma - \gamma_0)\Delta T \mu(æ_0 - 1)}{2\mu_0 + \mu(æ_0 - 1)}\frac{1}{z} + O\left(\frac{1}{|z|^2}\right).$$

The components of the vector $\mathscr{D}$ can be found as the coefficients multiplying the derivatives of the Green's tensor in the asymptotic expansion (14). After some routine calculations we obtain

$$\mathscr{D} = \frac{\mu\pi(\gamma - \gamma_0)\Delta T(æ_0 - 1)(æ + 1)}{\big(2\mu_0 + \mu(æ_0 - 1)\big)(æ - 1)}\begin{pmatrix}1\\1\\0\end{pmatrix}. \tag{25}$$

Two characteristics of the thermoelastic inclusion (Pólya-Szegö matrix $\mathscr{P}$ and vector $\mathscr{D}$) are necessary to determine the formula for the crack deviation due to the presence of the thermoelastic circular inclusion. The last can be written after certain simplifications in the following form

$$\Delta h(l) = \varepsilon^2\left(\Delta h_\varepsilon^\mu(l) + \frac{1}{K_\mathrm{I}}\Delta h_\varepsilon^T(l)\right) + O(\varepsilon^3), \tag{26}$$

where

$$\Delta h_\varepsilon^\mu(l) = \frac{1}{2}\frac{\mu_0 - \mu}{æ\mu_0 + \mu}\big(\cos\theta - (\cos\theta)^3\big) + \frac{1}{2}\frac{\mu_0(æ - 1) - \mu(æ_0 - 1)}{\mu(æ_0 - 1) + 2\mu_0}\big((\cos\theta)^2 + \cos\theta - 2\big),$$

$$\Delta h_\varepsilon^T(l) = \frac{\mu\sqrt{2\pi}(\gamma_0 - \gamma)\Delta T(æ_0 - 1)}{2\mu_0 + \mu(æ_0 - 1)}\int_0^\theta \frac{\sin\frac{3\phi}{2}}{\sqrt{\sin\phi}}\mathrm{d}\phi,$$

or in the dimensional form (when the distance between the center of the inclusion and the unperturbed crack y_0 is not normalized)

$$\Delta h(l) = \varepsilon^2\left(\Delta h_\varepsilon^\mu(l) y_0 + \frac{1}{K_\mathrm{I}}\Delta h_\varepsilon^T(l) y_0^{3/2}\right) + O(\varepsilon^3), \quad \varepsilon = \frac{R}{y_0}. \tag{27}$$

5.2. Interaction of a crack with elliptical thermoelastic inclusions

In this section another example is considered: the perturbation of a crack due to an elliptical thermoelastic inhomogeneity. An elliptical inclusion embedded in the elastic matrix has semi-axes a and b, with the major axis inclined at an angle β to x-axis. The problem of the crack-inclusion interaction reduces to finding the trajectory (22) and requires the values of the Pólya-Szegö matrix $\mathscr{P}$ and the vector $\mathscr{D}$ for a given inclusion. The components of the Pólya-Szegö (symmetric) tensor have been obtained by Movchan and Serkov (1997) (formulae (3.20) and (3.23) of that paper).

To find the crack trajectory in the closed form the thermo-elastic vector $\mathscr{D}$ for the elliptical inhomogeneity has to be evaluated. The algorithm includes solution of the boundary value problem for the vector field $\boldsymbol{W}^{(4)}$ (Eq. (13)) and extraction of the components of the vector $\mathscr{D}$ from the asymptotic expansion of the solution at infinity. The following representation for the displacement fields (inside and outside the defect) is used

$$\boldsymbol{W}_0^{(4)}(\boldsymbol{X}) = \boldsymbol{W}_0^{(4)*}(\boldsymbol{X}) + \alpha_0 \Delta T \boldsymbol{X}, \quad \boldsymbol{W}^{(4)}(\boldsymbol{X}) = \boldsymbol{W}^{(4)*}(\boldsymbol{X}) + \alpha \Delta T \boldsymbol{X}, \tag{28}$$

where $\boldsymbol{W}^{(4)}(\boldsymbol{X})$ are the displacement fields inside and outside the defect, α and α_0 are the coefficients of the thermal expansions. The fields $\boldsymbol{W}^{(4)*}(\boldsymbol{X})$ do not have the meaning of the total displacements, they are just the remaining terms after subtracting the linear thermal expansion fields. It can be easily checked that the boundary conditions in Eq. (13) can be rewritten for the fields $\boldsymbol{W}^{(4)*}(\boldsymbol{X})$ and $\boldsymbol{W}_0^{(4)*}(\boldsymbol{X})$ in the form

$$\boldsymbol{\sigma}_0^{(n)}\left(\boldsymbol{W}_0^{(4)*}; \boldsymbol{X}\right) = \boldsymbol{\sigma}^{(n)}\left(\boldsymbol{W}^{(4)*}; \boldsymbol{X}\right),$$

$$\boldsymbol{W}_0^{(4)*}(\boldsymbol{X}) - \boldsymbol{W}^{(4)*}(\boldsymbol{X}) = (\alpha - \alpha_0)\Delta T \boldsymbol{x}. \tag{29}$$

Instead of a jump in the traction conditions (13) we have a jump in the fields $\boldsymbol{W}^{(4)*}(\boldsymbol{X})$.The coefficients of the thermal expansion are related by $\gamma_i = 2(\mu_i + \lambda_i)\alpha_i$.

The problem we are considering is the boundary value problem (13) and it describes the fields induced by the defect on the crack faces. The crack propagates under the remote Mode-I loading (5) and the solution of Eqs. (13) and (29) should give the perturbation of the external loading only. Using the complex potential method the interface conditions (Eq. (29)) can be rewritten in the form:

$$\frac{1}{2\mu}\left[æ\varphi(\xi) - \frac{\omega(\xi)}{\omega'(\xi)}\overline{\varphi'(\xi)} - \overline{\psi(\xi)}\right] - \frac{1}{2\mu_0}\left[æ_0\varphi_0(\xi) - \frac{\omega(\xi)}{\omega'(\xi)}\overline{\varphi_0'(\xi)} - \overline{\psi_0(\xi)}\right]$$

$$= (\alpha_0 - \alpha)\omega(\xi)\Delta T,$$

$$\varphi(\xi) + \frac{\omega(\xi)}{\omega'(\xi)}\overline{\varphi'(\xi)} + \overline{\psi(\xi)} = \varphi_0(\xi) + \frac{\omega(\xi)}{\omega'(\xi)}\overline{\varphi_0'(\xi)} + \overline{\psi_0(\xi)}, \quad |\xi| = 1, \tag{30}$$

where the field

$$W^{(4)*}(z) = \frac{1}{2\mu}\left(æ\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\right)$$

satisfies the remote condition

$$W^{(4)*}(z) \to -\alpha \Delta T z, \quad z = X_1 + iX_2 \text{ as } |z| \to \infty.$$

The function $\omega(\xi)$ is the Zukovskij function

$$z = \omega(\xi) = c_1 \xi + c_{-1} \xi^{-1}, \quad \text{with} \quad c_1 = \frac{a+b}{2} \quad \text{and} \quad c_{-1} = \frac{a-b}{2} e^{2i\beta}. \tag{31}$$

The function (31) maps conformably the region corresponding to the elliptical inclusion in the z-plane with a slit along the major axis of the ellipse between the point $-e^{i\beta}\sqrt{a^2-b^2}$ and the point $e^{i\beta}\sqrt{a^2-b^2}$, to a ring in the ξ-plane of internal radius $\mathscr{R} = \sqrt{\frac{a-b}{a+b}}$ and of a unit external radius. So $0 \le \mathscr{R} \le 1$ and the limit values 0 and 1 correspond to a circular and a line inclusion, respectively. The same function (31) maps the region external to the ellipse into the region external to the circle of a unit radius. Solving the system of equation (30) by the Kolosov method the complex potentials outside the defect are found in the following form

$$\varphi(z) = -\frac{\gamma}{2}\Delta T z - \frac{1}{2z} \frac{(a+b)(\gamma_0 - \gamma)\Delta T(æ_0 - 1)\mu(\mathscr{R}^4 - 1)\mathscr{R}\Theta e^{i\beta}}{\left[(æ_0 - 1)\mu + 2\mu_0\right] + \mathscr{R}^4 \Theta\left[(æ_0 - 1)\mu - 2æ\mu_0\right]} + o\left(\frac{1}{z}\right),$$

$$\psi(z) = \frac{1}{2z} \frac{(a+b)(\gamma_0 - \gamma)\Delta T(æ_0 - 1)\mu(\mathscr{R}^4 - 1)(1 - \mathscr{R}^2\Theta)}{\left[(æ_0 - 1)\mu + 2\mu_0\right] + \mathscr{R}^4 \Theta\left[(æ_0 - 1)\mu - 2æ\mu_0\right]} + o\left(\frac{1}{z}\right), \tag{32}$$

That leads to the components of the vector $\mathscr{D}$ to be given by the following expression:

$$\mathscr{D}_1 = \frac{(a+b)\Lambda}{2(æ-1)}\left[1 - \mathscr{R}^2\Theta\left(\mathscr{R}^2 - (æ-1)\cos\beta\right)\right], \quad \mathscr{D}_2 = \frac{(a+b)\Lambda}{2(æ-1)}\left[1 - \mathscr{R}^2\Theta\left(\mathscr{R}^2 + (æ-1)\cos\beta\right)\right],$$

$$\mathscr{D}_3 = \frac{(a+b)\sqrt{2}\Lambda}{2}\mathscr{R}^2\Theta\sin\beta, \quad \Lambda = \frac{(\gamma_0 - \gamma)\Delta T(æ_0 - 1)\mu\pi(\mathscr{R}^4 - 1)(æ + 1)}{\left[(æ_0 - 1)\mu + 2\mu_0\right] + \mathscr{R}^4\Theta\left[(æ_0 - 1)\mu - 2æ\mu_0\right]}. \tag{33}$$

As before, we split up the total deflection of a crack on the elastic deflection (deflection under zero increment of temperature) and the thermal deflection (difference between total and elastic deflection). The elastic deflection can be specified by the following formula:

$$\Delta h(l)^{\mu} = \frac{ab}{4y_0}\Big\{\Omega(1 - \mathscr{R}^4\Theta)\left(t + t^2 - 2\right) + 2\mathscr{R}^2\Omega\Theta\Big[\sin 2\beta\left(t + t^2\right)(2t-1)\sqrt{1-t^2} - \cos 2\beta(t - t^3)(1+2t)\Big]$$

$$+\frac{\Theta}{1 + \mathscr{R}^4\Theta}(t - t^3)\Big[\left(1 + \mathscr{R}^4\Sigma\cos^2 2\beta\right)(1-t)(1+2t)^2$$

$$+\left(1 + \Sigma\sin^2 2\beta\right)(1+t)(2t-1)^2 - \mathscr{R}^4\Sigma\sin 4\beta\sqrt{1-t^2}(4t^2-1)\Big]\Big\} \quad t = \cos\theta \tag{34}$$

where

$$\Omega = \frac{2\big((æ-1)\mu_0 - (æ_0-1)\mu\big)}{\big((æ_0-1)\mu + 2\mu_0\big) + \mathscr{R}^4\Theta\big((æ_0-1)\mu - 2æ\mu_0\big)},$$

$$\Sigma = \frac{2\Theta(æ+1)\mu_0}{\big((æ_0-1)\mu + 2\mu_0\big) + \mathscr{R}^4\Theta\big((æ_0-1)\mu - 2æ\mu_0\big)}.$$

Whereas the thermal deflection is given by

$$\Delta h^T(l) = \frac{(a+b)\Lambda}{\sqrt{2\pi y_0}(æ+1)K_{\mathrm{I}}}\big\{(\mathscr{R}^4\Theta - 1)I_1(\theta) + \mathscr{R}^2\Theta\big(I_3(\theta)\sin\beta - I_2(\theta)\cos\beta\big)\big\}, \tag{35}$$

where I_1, I_2, I_3 are the integrals defined below:

$$I_1(\theta) = \int_0^\theta \frac{\sin\frac{3\phi}{2}}{\sqrt{\sin\phi}}\mathrm{d}\phi, \quad I_2(\theta) = \int_0^\theta \frac{\cos\phi\sin\frac{5\phi}{2} + \sin\frac{7\phi}{2}}{\sqrt{\sin\phi}}\mathrm{d}\phi, \quad I_3(\theta) = \int_0^\theta \frac{\cos\phi\cos\frac{5\phi}{2} + \cos\frac{7\phi}{2}}{\sqrt{\sin\phi}}\mathrm{d}\phi.$$

One can observe that the elastic deflection of the crack at infinity turns out to be the same as at the origin, and it can be evaluated using the substitution $\theta = \pi$ and $\theta = \pi/2$ in Eq. (34):

$$\Delta h^\mu_\infty = \Delta h^\mu(0) = \frac{ab}{2y_0}\Omega(\mathscr{R}^4\Theta - 1) \tag{36}$$

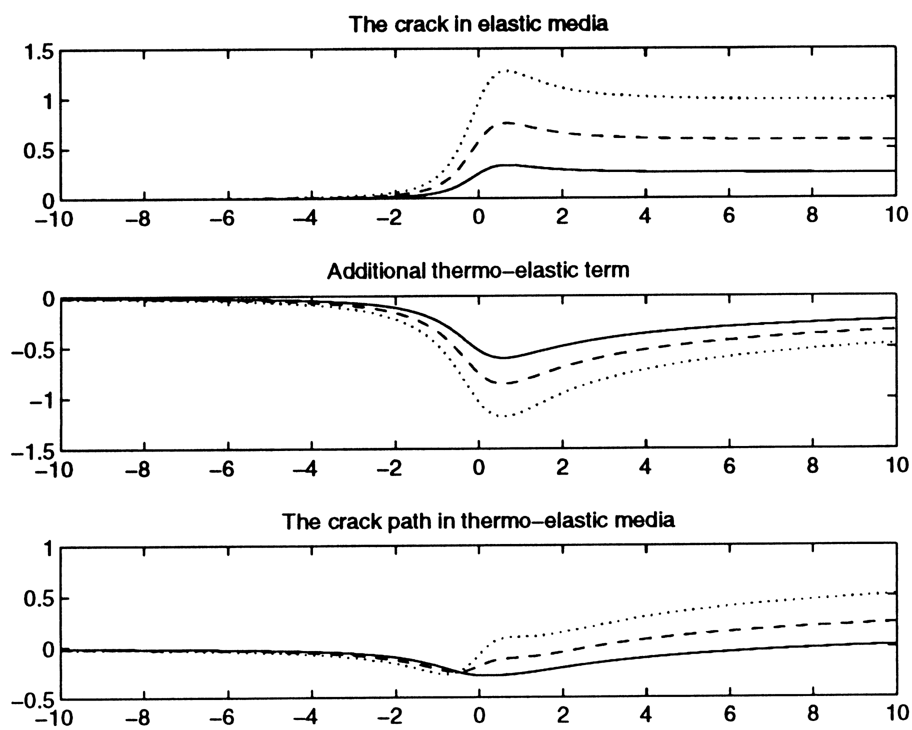

Fig. 2. The crack deviation due to the soft thermoelastic inclusion: $\gamma = 4\cdot 10^{-3}$, $\gamma_0 = 10^{-3}$, $\Delta T = 100$ K, $æ = 2$, $æ_0 = 2$, $\mu_0/\mu = 0.5(-)$, $\mu_0/\mu = 0.2(--)$, $\mu_0/\mu = 0.01(\ldots)$.

At the same time, the contribution of the thermal deflection at infinity is always equal to zero:

$$\Delta h^T \to \frac{(a+b)\Lambda}{\sqrt{2\pi y_0}(æ+1)K_\mathrm{I}}\Big\{(\mathscr{R}^4\Theta - 1)I_1(\pi) + \mathscr{R}^2\Theta\big(I_3(\pi)\sin\beta - I_2(\pi)\cos\beta\big)\Big\} = 0.$$

Thus, the thermal deflection has only local effect, it affects the trajectory near the inclusion and does not affect it at remote points.

5.3. Numerical data for the crack trajectories

Crack trajectories in the elastic media have been analyzed previously by Valentini et al. (1999) for inclusions with ideal interphase contact and by Bigoni et al. (1998) for defects with debonding (imperfect interphase) conditions. In this section, we describe the numerical results for analysis of the inhomogeneous medium with a crack under additional heating (or cooling).

At the beginning we recall some results for the inhomogeneous elastic materials (no temperature drop). Namely, the cavities and soft inclusions attract the crack, and the inclusions, which are more rigid than the matrix material, repel the crack. If one introduces an additional temperature field, these statements become no longer valid. Depending on the temperature increment the crack deflection can either be positive or negative for a range of the elastic parameters of the inclusions.

In numerical experiments the results of which are described below the trajectories of the cracks are

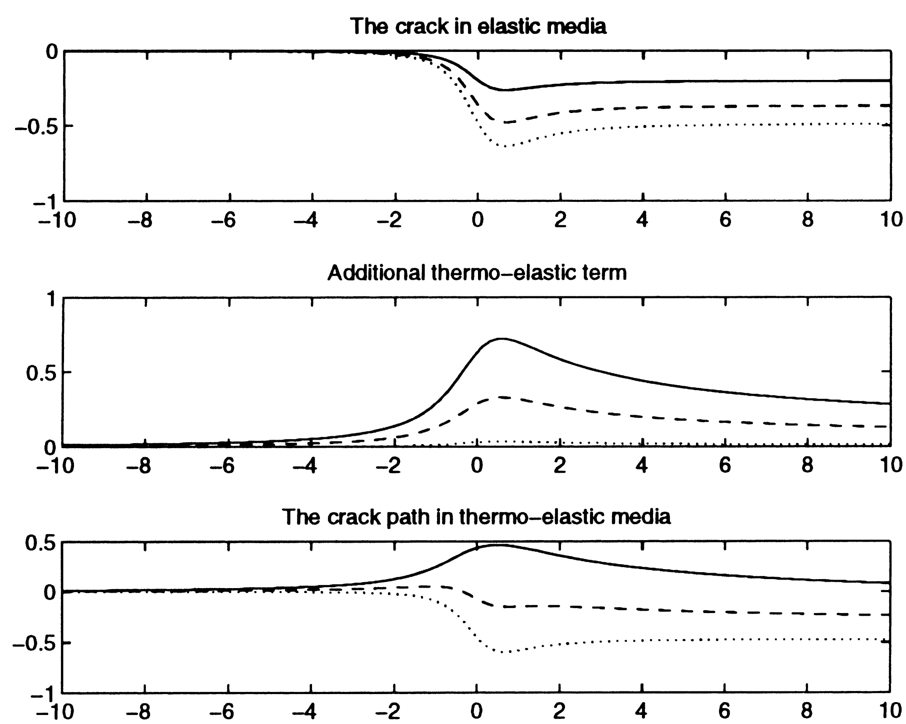

Fig. 3. The crack deviation due to the rigid thermoelastic inclusion: $\gamma = 10^{-3}$, $\gamma_0 = 10^{-2}$, $\Delta T = 100$ K, $æ = 2$, $æ_0 = 2$, $\mu_0/\mu = 2(-)$, $\mu_0/\mu = 5(--)$, $\mu_0/\mu = 50(\ldots)$.

split into the 'elastic' trajectories (trajectories of a crack in a medium with zero increment of a temperature) and the thermal trajectories (difference between total and elastic deflection functions).

In Fig. 2 (top plot) the crack trajectories in a medium with soft inclusions are given. The deflection functions correspond to the crack being attracted by the same size inclusions with different shear moduli. At the infinitely remote point, as well as in all other points of the crack trajectory, the deflection is positive, it corresponds to an attraction of the crack by the inclusion. However the presence of the temperature (middle plot) changes the situation. It reduces the deflection of the crack near the inclusion (bottom plot) and for some parameters the trajectory can be almost a straight line (solid line, bottom plot). This example shows that the temperature can smooth out the crack path and change the positive deflection by the negative (at least on the part of the trajectory).

In Fig. 3 (top plot) the crack trajectories caused by rigid inclusions are shown. Negative deflection in all points of trajectories including the infinite point is the characteristic feature of the rigid defects. Than more rigid the inclusion is than it repels the crack more. It is important to note that all possible crack trajectories are in the region bounded by the curve corresponding to $\mu_0 \to \infty$ (lower bound) and $\mu = 0$ (upper bound). Here the temperature acts in a different way, it provides the additional positive deflection in such a way that the total deflection can be either positive or negative. Again we can see that under certain values of parameters the amplitude of the deflection nearly vanishes (dashed line, bottom plot).

In Fig. 4 (top plot) the case of different Poisson's ratios is considered. Shear moduli of the matrix and the inclusion are equal, but the parameters æ and æ$_0$ which characterize the Poisson's ratios vary. If the

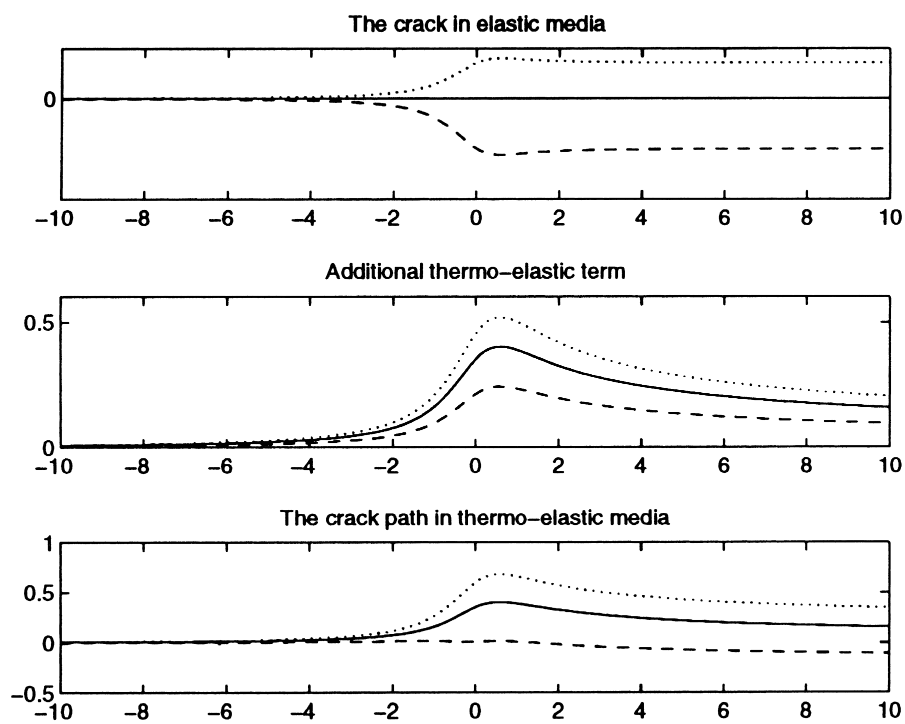

Fig. 4. The crack deviation due to the thermoelastic inclusions with the same shear modulus: $\gamma = 10^{-3}$, $\gamma_0 = 4 \cdot 10^{-3}$, $\Delta T = 100$ K, æ $= 2$, $\mu_0/\mu = 1$, æ$_0 = 2(-)$, æ$_0 = 1.5(--)$, æ$_0 = 2.5(\ldots)$.

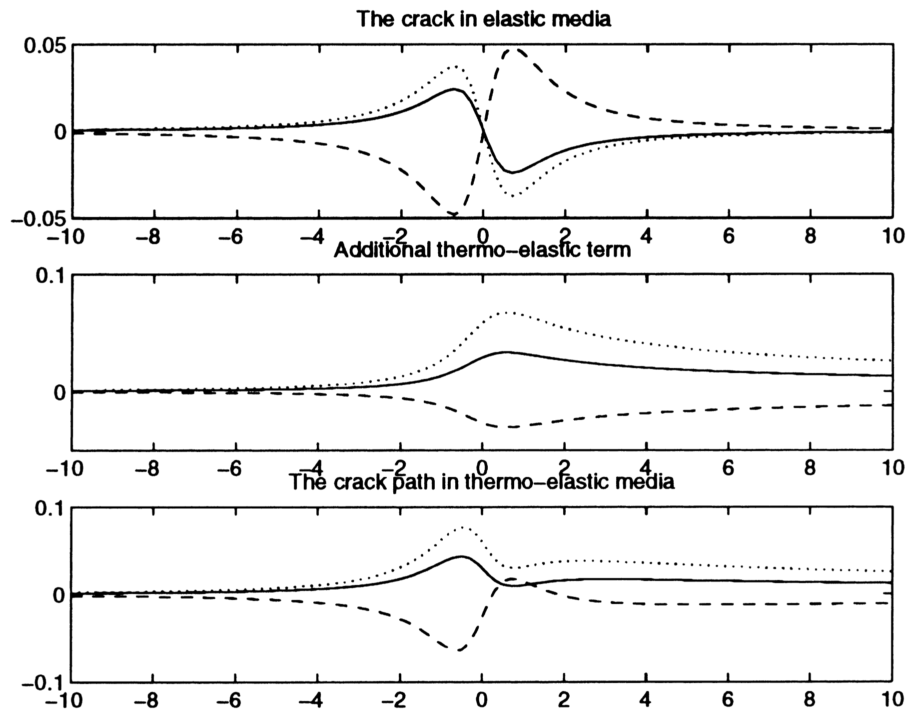

Fig. 5. The crack deviation due to the thermoelastic inclusions with the same bulk modulus: $\gamma = 10^{-3}$, $\Delta T = 25$ K, $\mu = 1$, $\lambda = 1$, $\mu_0 = 1.5$, $\lambda_0 = 0.5$, $\gamma_0 = 2 \cdot 10^{-3}(-)$, $\mu_0 = 0.5$, $\lambda_0 = 1.5$, $\gamma_0 = 10^{-4}(--)$, $\mu_0 = 1.95$, $\lambda_0 = 0.05$, $\gamma_0 = 3 \cdot 10^{-3}(\ldots)$.

Poisson's ratio of the inclusion is less than the Poisson's ratio of the matrix the crack is attracted by the inclusion. In contrary, if Poisson ratio of the inclusion is greater the crack is repelled by the inclusion. If the Poisson's ratios of the matrix and the inclusion are equal (and there is ideal interface contact on the border) there is no any crack deflection, the trajectory is a straight line (solid line, top plot). However that is true for zero temperature drop only. Once the elastic plane is under cooling (heating) the crack perturbs (solid line, bottom plot). In contrary, the elastic deflection can be non-zero for zero temperature change (dashed line, top plot) and the trajectory becomes straight if the temperature field is imposed (dashed line, bottom plot).

In further analysis the cracks with zero deflection at infinity are considered. There are special types of defects which the macro-crack is not sensitive to. Perturbation caused by these defects has only the local effect — it occurs near the defect only. Any perturbations vanish when crack propagates further. Such types of defects have the same bulk modulus as the material of the matrix. In Fig. 5 (top plot) the illustration of this effect is presented: crack deflection at infinity is zero and there are local perturbations near the origin. The last are caused by the difference in shear moduli of the matrix and the inclusion. The perturbation changes sign from positive to negative (or vice versa) depending on either the shear modulus greater in the inclusion or in the matrix.

The thermal deflection itself (difference between the total deflection and the elastic one) is caused by the thermal expansion effects only. If, for example, the coefficients of bulk thermal expansion γ and γ_0 are equal then the thermal deflection disappears even for a non-zero temperature increment. The integral effect of the temperature can be specified by the fact either the inclusion repels the crack or attracts it. It depends on the difference between the thermal expansion coefficients of the inclusion and the matrix and

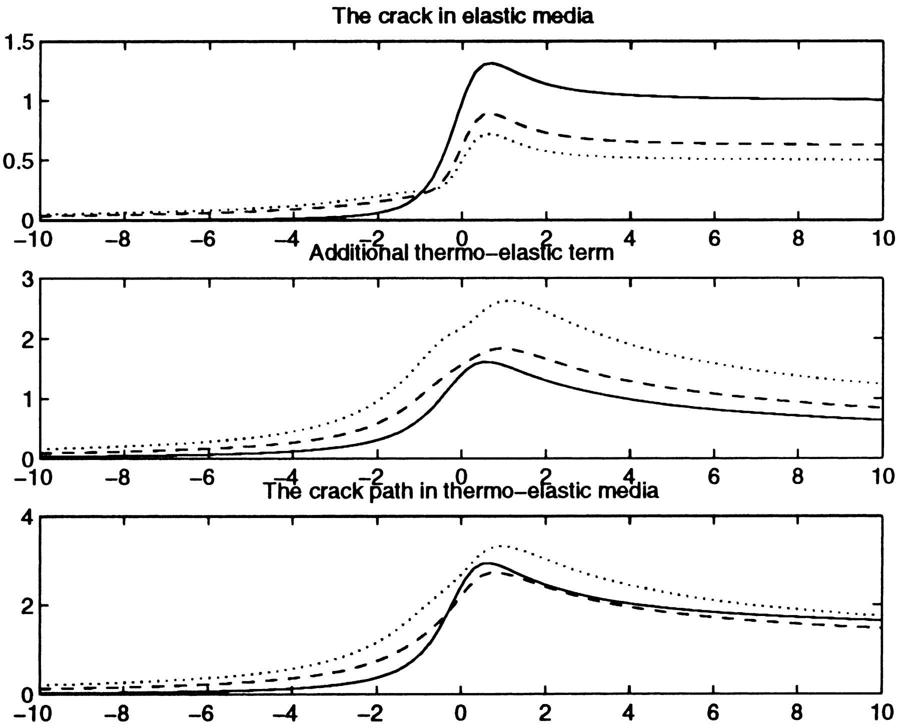

Fig. 6. The crack deviation due to the elliptical thermoelastic inclusions with the same elastic and thermal moduli: $\gamma = 10^{-3}$, $\gamma_0 = 5 \cdot 10^{-3}$, $\Delta T = 100$ K, æ = 2, $\mu_0 = 0$, $\mu = 1$, $æ_0 = 2$, $\beta = 45°$, $a = 1$, $b = 1(-)$, $b = 0.5(--)$, $b = 0(\ldots)$.

the sign and absolute value of the temperature drop. The inclusion with greater thermal expansion coefficient then of the matrix repels the crack (the temperature change is positive). In contrary, the inclusion with a small thermal expansion coefficient attracts the crack. This effect can be formulated in the following statement:

The sign of the thermal deflection of the crack is the same as the sign of $(\gamma_0 - \gamma)\Delta T$.

The fact is illustrated in Figs. 2 and 3. The middle plots show the thermal crack trajectory is the negative function for any variety of elastic parameters provided $\gamma > \gamma_0$, $\Delta T > 0$. For $\gamma < \gamma_0$, $\Delta T > 0$ the thermal deflection is positive at any point of the crack trajectory.

The physical process is described by the total deflection (bottom plot in Figs. 2–5) where the elastic and temperature effects are taken into account both. One of the results is that there are situations when the temperature term compensates the influence of the elastic term, and the resulting crack deflection decreases (or nearly vanishes). The examples are presented in Figs. 2 and 5 (bottom plots).

Despite the fact that the analysis of the crack trajectory for an elliptical inclusion is more complicated and more parameters involved (eccentricity m, rotation angle β) its integral impact on the crack trajectory is similar one of the circular defect. In Fig. 6 we give the example of crack deflection due to the elliptical elastic inclusions of different eccentricities. It shows that crack trajectories can have different shapes depending on the parameter b, but integral effects (either deflection is positive or negative or zero) do not depend on elongation of the ellipse. Another interesting observation is that the rotation of the ellipse does not affect the value of deflection at infinitely remote point (it follows from Eq. (36)).

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References

Bigoni, D., Movchan, A., Esposito, L., Serkov, S., Valentini, M., 1996. Crack propagation in brittle, elastic solids with defects. Atti del: XII Convegno Nazionale del Gruppo Italiano di Frattura, Parma, pp. 59–67.

Bigoni, D., Serkov, S.K., Valentini, M., Movchan, A.B., 1998. Asymptotic models of dilute composites with imperfectly bounded inclusions. International Journal of Solids and Structures 35 (24), 3239–3258.

Bueckner, H.F., 1970. A novel principle for the computation of stress intensity factors. ZAMM 50 (9), 529–546.

Eshelby, J.D., 1957. The determination of the elastic fields of an ellipsoidal inclusion, and related problems. Proceedings of the Royal Society London A 241, 376–396.

Maz'ya, V.G., Nazarov, S.A., Plamenevski, B.A., 1992. Asymptotische Theorie Elliptischer Randwertanfgaben in Singulär Gestörten. Akademie Verlag, Berlin.

Movchan, A.B., Serkov, S.K., 1997. The Pólya-Szegö matrices in asymptotic models of dilute composites. European Journal of Applied Mathematics 8, 595–621.

Movchan, A.B., Willis, J.R., 1995. Dynamic weight functions for a moving crack. II Shear loading. Journal of the Mechanics and Physics of Solids 43 (9), 1369–1383.

Muskhelishvili, N.I., 1963. Some Basic Problems of the Mathematical Theory of Elasticity. Noordhoff, Groningen.

Rubinstein, A.A., 1986. Macro-crack–micro-defect interaction. Journal of Applied Mechanics 53, 505–510.

Rubinstein, A.A., Choi, H.C., 1988. Macrocrack interaction with transverse array of microcracks. International Journal of Fracture 36 (1), 15–26.

Sih, G.C. (Ed.), 1973, 1975, 1977, 1978. Mechanics of Fracture, vols. 1–5. Noordhoff, Leiden.

Sih, G.C., 1991. Mechanics of Fracture Initiation and Propagation. Kluwer Academic Publishers, Dordrecht, Boston, London.

Sih, G.C., Chen, E.P., 1981. Mechanics of Fracture. Cracks in Composite Materials, vol. 6. Martinus Nijhoff Publishers, Hague, Boston, London.

Timoshenko, S.P., Goodier, J.N., 1951. Theory of Elasticity. McGraw-Hill, New York.

Valentini, M., Serkov, S.K., Bigoni, D., Movchan, A.B., 1999. Crack propagation in a brittle elastic material with defects. Journal of Applied Mechanics 66, 79–86.

Willis, J.R., Movchan, A.B., 1995. Dynamic weight-functions for a moving crack. I Mode-I loading. Journal of the Mechanics and Physics of Solids 43 (3), 319–341.

Willis, J.R., Movchan, A.B., 1997. Three-dimensional dynamic perturbation of a propagation crack. Journal of the Mechanics and Physics of Solids 45 (4), 591–610.

Zorin, I.S., Movchan, A.B., Nazarov, S.A., 1988. Use of the elastic polarization tensor in problems of crack mechanics. Izv. AN SSSR. Mechanika Tverdogo Tela 23 (6), 128–134.